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P.I. G-RINGS AND THE CONTRACTABILITY OF PRIMES

BY

AMIRAM BRAUN

ABSTRACT

Let $R = F\{x_1, \cdots, x_k\}$ be a prime affine p.i. ring and S a multiplicative closed set in the center of R, $Z(R)$. The structure of G-rings of the form R_S is completely determined. In particular it is proved that $\overline{Z(R_S)}$ — the normalization of $Z(R_S)$ — *is a prüfer ring*, $1 \leqq \text{k.d}(R_S) \leqq \text{p.i.d}(R_S)$ and the inequalities can be strict. We also obtain a related result concerning the contractability of q, a prime ideal of $Z(R)$ from R. More precisely, let Q be a prime ideal of R maximal to satisfy $Q \cap Z(R) = q$. Then $\text{k.d}\, Z(R)/q = \text{k.d}\, R/Q$, $h(q) = h(Q)$ and $h(q) + \text{k.d}\, Z(R)/q = \text{k.d}\, z(R)$. The last condition is a necessary but *not* sufficient condition for contractability of q from R.

Introduction

Given $R = F\{x_1, \cdots, x_k\}$, a prime affine p.i. ring, one of our main purposes is to characterize the G-rings of the form R_S (R localized by S, where $0 \not\in S$ is a multiplicative closed set of $Z(R)$, the center of R). By a G-ring we mean a prime ring such that the intersection of all non-zero ideals is non-zero. We get the following theorem.

THEOREM. *Let $R = F\{x_1, \cdots, x_k\}$ be a prime affine* p.i. *ring, R_S — a G-ring where $S \subset Z(R)$, a multiplicative closed set. Then*

(1) *$Z(R_S)$, R_S, have a* finite *number of prime ideals.*

(2) *Each $q \subset Z(R_S)$, a prime ideal, is contracted in an isolated fashion from R_S.*

(3) $\text{k.d}(R_S) \leqq \text{p.i.d}(R_S) = n$ ($\text{k.d}(R_S) \equiv \text{Krull. dim.}\,(R_S)$).

(4) *For each prime ideal q of $Z(R_S)$, q_q is finitely generated and if q is maximal q is finitely generated.*

(5) *$\overline{Z(R_S)}$, the normalization of $Z(R_S)$, is a* prüfer *ring.*

(6) *Each finitely generated ideal in $\overline{Z(R_S)}$ is generated by $n+1$ (or less) elements.*

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Unlike the noetherian commutative case it might happen that $\mathrm{k.d}(R_S) > 1$. An example (essentially due to G. Bergman) illustrates it.

A related question that we try to handle is the following. Given $q \lhd Z(R)$, $R = F\{x_1, \cdots, x_k\}$ an affine prime p.i. ring, is there a prime ideal Q in R such that $Q \cap Z(R) = q$? Examples due to Rowen [9] and L. Small (unpublished) show that this is not always the case. On the other hand if there exists $p \subsetneqq q$, p a prime ideal of $Z(R)$, $P \subset R$, a prime ideal of R with $P \cap Z(R) = p$, $\mathrm{p.i.d}(P) = \mathrm{p.i.d}(R)$ and $h(q/p) = 1$, then it is well known that q is contractable. Again this is not always the case, in fact, in trying to approximate a contractable prime q from below, the $R_S - G$ rings occur. Anyhow certain properties of contractable primes are obtained, in fact we have the following

THEOREM. *$R = F\{x_1, \cdots, x_k\}$ is an affine prime* p.i. *ring, $q \subset Z(R)$ a prime ideal. Let $Q \subset R$ be a prime ideal, maximal with respect to $Q \cap Z(R) = q$. Then*

(1) $\mathrm{k.d.}\ Z(R)/q = \mathrm{k.d}\, R/Q$.

(2) $h(q) = h(Q)$.

(3) $\mathrm{k.d.}\, Z(R)/q + h(q) = \mathrm{k.d}\, Z(R)$.

One can regard (3) of the last theorem as a necessary condition for contractability of a prime ideal q in $Z(R)$. This condition is not sufficient as the example of Rowen [9] shows.

As for notations, we mainly use the same as in [1]. Let us mention some. We write $q \lhd C$ meaning that q is an ideal of C. We always write $C \equiv Z(R)$ where R is an affine prime p.i. ring. $\overline{Z(R)} =$ the integral closure of $Z(R)$ in its quotient field. We also use the canonical isomorphism $R_q/P_q \cong (R/P)_q$. Here $q \lhd C$ is prime and R_q means the localization of R with respect to $C - q$. Now $(R/P)_q$ means a localization of R/P with respect to $S = C - q + P/P$ in $Z(R/P)$. S is a multiplicatively closed set if there exists a $Q \lhd R$, prime with $Q \supset P$ and $Q \cap C = q$, and we will use it under such assumptions. We have cause for using $T(R)$ — the trace envelope of R, $T(R) \equiv R[c_1, \cdots, c_e]$ where c_i, $i = 1, \cdots, e$ are the coefficients of the characteristic polynomials of all monomials in $x_1, \cdots, x_k$ of degree $\leqq n^2$ ($n = \mathrm{p.i.d}\ (R)$). $T(R)$ is finite over its center. We use $h(q) \equiv$ height(q) — the maximal length of chains of prime ideals descending from q. The following commutative conventions are used: $\mathrm{k.d}(R)$ stands for the classical Krull dimension of R, and by "f.g. module" we mean a finitely generated module.

Finally R — a prime p.i. ring with $\mathrm{k.dim}\, R \geqq 1$ is said to be a G-ring if there exists $x \in Z(R)$ with $R[1/x]$ a simple (artinian) ring. This is equivalent to the fact that the intersection of all non-zero ideals in R is non-zero. (See Kaplansky's *Commutative Rings* for the Commutative Analogue.)

The paper is organized as follows. §1 is purely commutative and of independent interest (we think) the result of which is used in §3. In §2 we prove theorems on contractability as well as related results. In §3 we complete the proof of the structure of G-rings.

§1. All rings are commutative

The main result here is the following

THEOREM 1.1.[†] *Let C be a commutative domain* (*not* necessarily noetherian) *satisfying the following assumptions*:

(1) *C has finitely many prime ideals*;

(2) *for all $p \lhd C$, prime p_p is a* f.g. *ideal in C_P.*

Then $\bar{C}$ is a prüfer ring.

PROOF. We prove the theorem via induction on k.d(C), and without loss of generality we may assume that C is local with maximal ideal m. By induction we have that $\bar{C}_p$ is prüfer for every prime ideal $p \subsetneqq m$. Let $x \in C$ with $\mathrm{rad}(xC) = m$ and let $T \equiv C[1/x] \cap \bar{C}$. We shall show that in T every maximal ideal is invertible. We firstly show that T/xT is a finite $C/xT \cap C$ module; the argument we use is essentially the one in [5], [11]. Observe that C/xC, $C/xT \cap C$ are artinian, k.d. $(T/xT) = 0$ and every maximal in T contains x; we have that $x^eT \cap C \subseteq x^{e+1}T \cap C + xC$ for some e by the artinian property of C/xC. It suffices to show that $T/xT \subseteq x^{-e}C + xT/xT$ since the latter is a f.g. module over $C/xT \cap C$ hence artinian. Let $d \in T$, then $d \cdot x^n \in C$ for some $n = n(x)$ and we can take $n > e$. Now $x^n \cdot d \in x^nT \cap C$ hence $x^nd. = x^{n+1}d_1 + xd_2$ where $d_1 \in T$, $d_2 \in C$ hence $d \in xT + x^{-(n-1)}C$ and after $n - e$ steps we get that $d \in xT + x^{-e}C$ hence $T/xT \subset x^{-e}c + xT/xT$. This implies that every maximal ideal in T is f.g. We next show that each maximal q in T is invertible. Indeed, let $q^{-1} \cdot q = q$ ($q^{-1} \neq T$) and $a \in q^{-1}$, then $a \in \bar{C}$ since q is a f.g. ideal, also $ax \in aq \subseteq q$ hence $(ax)x^n \in C$ for some n, hence $a \in C[1/x]$ and consequently $a \in C[1/x] \cap \bar{C} = T$, a contradiction. Thus $q^{-1} \cdot q = T$.

We now show that $\bar{T} = \bar{C}$ is prüfer. We may assume that T is local with maximal m'. We also have that $\bar{T}_p$ is prüfer for every prime $p, p \subsetneqq m'$. We recall some standard facts: given $a, b \in T$, let $H(a,b) \equiv \{f(x,y) \in T[x,y] \mid f(at,bt) = 0\}$ where x, y, t are variables and let $C(H) =$ the ideal in T generated by the coefficients of the elements in $H(a,b)$; *then $\bar{T}$ is prüfer iff $C(H(a,b)) = T$ for every $a, b \in T$* [2]. Let $H \equiv H(a,b)$ one may easily

[†] I would like to thank W. Vasconcelos for improving an earlier version of this result.

check that $C(H_p) = C(H)_p$ where $p \lhd T$ and prime; it is also clear that H is a prime ideal in $T[x, y]$. Let $p \subsetneqq m'$ be a prime ideal; then $\bar{T}_p$ is prüfer hence $C(H_p) = C(H)_p = T_p$ and consequently $C(H) \not\subset p$. Thus if $C(H) \neq T$ we have that $\mathrm{rad}(C(H)) = m'$. Next one observes that $C(H)^{-1}C(H) \equiv C(H)$. Indeed let $d \in C(H)^{-1} \subset K$, where K is the q. field of T, and $\lambda = \Sigma C_{ij}x^iy^j \in H$ then $\Sigma C_{ij}(at)^i(bt)^j = 0$ hence $\Sigma(dC_{ij})(at)^i(bt)^j = 0$, but $dC_{ij} \in T$ hence $\Sigma dC_{ij}x^iy^j \in H$. Consequently $C(H)^{-1}C(H) = C(H)$, $(m')^{-1}C(H) = C(H)$. But $(m')^{-1}m' = T$ hence $C(H) = m'C(H)$; but $C(H)$ is f.g. and we get a contradiction via Nakayama. Thus $C(H) \not\subset m'$, that is, $C(H) = T$. Q.E.D.

REMARK 1.2. In an earlier version of this theorem we imposed an additional assumption on C, namely, given $q, p \lhd C$, primes with $q \subseteq p$ and $\mathrm{k.dim}(C/q)_p = 1$ then $(C/q)_p$ is Japanese, and we got the following additional conclusion: *$\bar{C}$ is a finite module over C* and in particular for every $p \lhd \bar{C}$, prime, p_p is a f.g. ideal.

§2. All rings are p.i. rings

In this section $R = F\{x_1, \cdots, x_k\}$ is a f.g. prime p.i. ring ($\equiv$ affine). $S \subset Z(R)$ will always denote a multiplicatively closed set, $0 \not\in S$, and R_S is the localization obtained by inverting S.

LEMMA 2.1. *Let R_S be a G-ring (R, S are as mentioned above). Then R_S, $Z(R_s)$ have finitely many prime ideals and every prime in $Z(R_S)$ is contracted in an isolated fashion from a prime in R_S.*

PROOF. R_S being a G-ring implies that there are only finitely many height one primes in R_S since there exists $\gamma \in \cap P$, $\gamma \neq 0$, $\gamma \in Z(R)$ and the intersection is on all primes of R_S, but the minimal primes above γR_S are finite in number since the same is true for γR in R [7, p. 107]. We may therefore assume that $\dim R_S > 1$. Say $k \geqq 2$ is the first integer with infinitely many primes of height k. There exists P_{k-1}, a prime ideal in R_S, with $h(P_{k-1}) = k-1$ and P_{k-1} is contained in infinitely many primes of height k. Let $P_{k-2} \subsetneqq P_{k-1}$, prime, we denote $R_S/P_{k-2} \equiv R^*_S$, $P_{k-1}/P_{k-2} = P^*_{k-1}$ and assume that $h(P^*_{k-1}) = 1$. By [12, lemma 4] there exists a prime P'_{k-1} in $T(R^*_S)$ with $R^*_S \cap P'_{k-1} = P^*_{k-1}$. We have $R^*_S/P^*_{k-1} \subset T(R^*_S)/P'_{k-1}$ and by [7, p. 110] all but a finite number of the infinitely many height one primes in R^*_S/P^*_{k-1} are contracted from $T(R^*_S)/P'_{k-1}$. Let $P^* \supset P^*_{k-1}$ prime with $h(P^*) = h(P^*_{k-1}) + 1$ and $P^1 < T(R^*_S)$ prime with $P^1 \supsetneqq P'_{k-1}$, $P^1 \cap R^*_S = P^*$. Then $h(P^1) > 1$ and there are infinitely many primes $\{W_\alpha\}$ in $T(R^*_S)$ with $h(W_\alpha) = 1$ and $W_\alpha \subsetneqq P^1$ (by the principal ideal theorem in $T(R^*_S)$). We have that $h(W_\alpha \cap R^*_S) = 1$ for all *but* finitely many α's. Indeed, if

$W_\beta = W$ with $h(W \cap R_S^*) > 1$ then $W \cap R_S^* = P^*[(h(P^*) = 2)$ and $W \cap R_S^* \subset P^*]$ but then W is a minimal prime over $P^*T(R_S^*)$ in $T(R_S^*)$ which is noetherian. By the same argument $\{W_\alpha \cap R_S^*\}$ is an *infinite* family of ideals. Let $V_\alpha \lhd R_S$ prime with $V_\alpha^* = W_\alpha \cap R_S^*$, then $V_\alpha \subsetneq P$ since $V_\alpha^* \subsetneq P^*$, $V_\alpha \supsetneq P_{k-2}$ but $h(P) = h(P_{k-1}) + 1 = k$ hence $h(V_\alpha) = k - 1$, a contradiction to the minimality of k.

We next show that each prime p in $Z(R_S)$ is contracted from R_S and hence there are finitely many primes in $Z(R_S)$. We argue by induction on $h(p)$. The case $h(p) = 1$ is well known (e.g., [1]). Say $h(p) = k$, by induction there are only finitely many primes $p_1, \cdots, p_r \subsetneq p$. Let $x \in p \setminus (p_1 \cup \cdots \cup p_r)$ then $xR_S \subsetneq R_S$ (otherwise x is a unit). Let $\mathrm{rad}(xR_S) = V_1 \cap \cdots \cap V_l$ then $(V_1 \cap Z(R_S))^{e_1} \cdots (V_l \cap Z(R_S))^{e_l} \subseteq xR_S \cap Z(R_S) = xZ(R_S) \subset p$ hence $p = V_j \cap Z(R_S)$ for some j. Finally, each prime in $Z(R_S)$ is contracted in an isolated fashion from one in R_S. This is achieved by Lemma 2.3 and an easy induction on the height of primes in $Z(R_S)$.

REMARK 2.2. Via the same lines one can prove the same for $R = \Lambda\{x_1, \cdots, x_k\}$ where Λ is merely a central noetherian domain and R is a prime p.i. ring.

LEMMA 2.3. *Let $p \lhd C$, $P \lhd R$, prime ideals with $P \cap C = p$ and P is maximal with respect to this property. Suppose $Q_1 \supset Q_2 \supsetneq P$ are primes with $Q_1 \cap C = Q_2 \cap C = q$, satisfying*

(1) *$h(Q_2) = h(P) + 1$,*

(2) *$(R/P)_q$ is a G-ring.*

Then $Q_1 = Q_2$.

PROOF. Let $(R/P)_q \equiv R^* \simeq R_q/P_q$, $C_q/P_q \cong (C/P)_q \equiv D$, we have $D \subset R^*$. By the maximality of P and (2) we have that R^* is a G-ring, $\mathrm{k.d}(R^*) \geqq 1$, hence $T(R^*)$ is a G-ring, but being noetherian implies that $\mathrm{k.d}\, T(R^*) = 1$.

We need the following observation: Let $v \lhd \overline{Z(T(R^*))}$, prime and $v \cap D' = w$, where D' is the integral closure of D in $\overline{Z(T(R^*))}$. We have that $\mathrm{k.d}\, \overline{Z(T(R^*))} = \mathrm{k.d}\, T(R^*) = 1$ hence by Zarisky's Main Theorem $h(w) = 1$ and $D'_w = \overline{Z(T(R^*))}_v$ is a D.V.R.

Continuing the proof let $Q_1^* = (Q_1/P)_q$, $Q_2^* = (Q_2/P)_q$, then $h(Q_2^*) = 1$. R^*D' is a central integral extension of R^*, hence we have by "Going Up" (e.g., [10]) prime ideals in R^*D', $Q_1' \supset Q_2'$ with $h(Q_1') = h(Q_1^*)$, $h(Q_2') = h(Q_2^*)$ and $Q_1' \cap R^* = Q_1^*$, $Q_2' \cap R^* = Q_2^*$. Let Q be a prime ideal in $T(R^*)\overline{Z(T(R^*))}$ with $Q \cap R^*D' = Q_2'$, $h(Q) = 1$ (height one in R^* is contracted from one in $T(R^*)$ hence one in $T(R^*)\overline{Z(T(R^*))}$, e.g., [12]). Let $v = Q \cap \overline{Z(T(R^*))}$ and

$w = v \cap D'$, then $Q_2' \cap D' = w$. Let $Q_1' \cap Z(R^*D') = t$ and $Q_2' \cap Z(R^*D') = s$, then $t \supseteq s$. We have the following inclusions:

$$\begin{array}{ccccccc} D' \subseteq Z(R^*D') & \subset & R^*D' & \subset & T(R^*)Z(T(R^*)) \\ \cup & & \cup & & \\ t & & Q_1' & & \\ \cup & & \cup & & \\ w \subset s & \subset & Q_2' & \subset & Q \end{array}$$

Now, $t \cap D' = s \cap D'$ since D' is integral over D and $(q/p)_q = (t \cap D') \cap D = (s \cap D') \cap D$; but $s \cap D' = w$ and D'_w is a D.V.R. by the previous observation, hence $D'_w = Z(R^*D')_w = Z(R^*D')_t$, $h(t) = 1$ and $s = t$. But R^*D' is a localization of an integral extension of an affine p.i. ring and Krull dim $Z(R^*D')_t = 1$, by [1] we get that $h(Q_1') = h(Q_2')$ and consequently $h(Q_1^*) = h(Q_2^*) = 1$. Q.E.D.

THEOREM 2.4. *Let $R = F\{x_1, \cdots, x_k\}$ be a prime* p.i. *ring, $q \lhd C$ prime. Let $Q \lhd R$ be prime with Q being maximal to satisfy $Q \cap C = q$. Then $h(q) \leqq h(Q)$ and* k.dim C/q = k.dim R/Q.

PROOF. Let $q_1 \supset q$, $q_1 \lhd C$ be a minimal among the prime ideals of C such that there exists $Q_1 \lhd R$, prime, $Q_1 \supset Q$ and $Q_1 \cap C = q_1$. If $q_1 = q$ then by the choice of Q, *Q is a maximal ideal in R* hence q is maximal in C [7, p. 102] hence $h(q) \leqq$ k.dim C = k.dim $R = h(Q)$. We may assume that $q_1 \supsetneqq q$ and further assume that $h(Q_1) = h(Q) + 1$ (again by the choice of Q and q_1). We have that $(C/q)_{q_1} \cong C_{q_1}/q_{q_1} \subset R_{q_1}/Q_{q_1} \cong (R/Q)_{q_1}$. The assumptions above and the minimality of q_1 imply that every non-zero ideal prime ideal of $(R/Q)_{q_1}$ contracts to $(q_1/q)_{q_1}$, hence $(R/Q)_{q_1}$ is a G-ring, and by Lemma 2.3 we have that Krull dim$(R/Q)_{q_1}$ = 1 and if $Q' \supset Q$, prime with $Q' \cap C = q_1$ then $h(Q') = h(Q) + 1$, thus (Q_1, q_1) satisfies the maximality property. We continue the process with (q_1, Q_1), by choosing $q_2 \supset q_1$, $q_2 \lhd C$ minimal above $q_1 \cdots$. We get the following chain of prime ideals $q_0 = q \supsetneqq q_1 \supsetneqq q_2 \cdots \supsetneqq q_m$, $Q_0 = Q \supsetneqq Q_1 \supsetneqq Q_2 \cdots Q_{m-1} \supsetneqq Q_m$, $q_i \lhd C$, $Q_i \lhd R$ are prime ideals for $i = 0, \cdots, m$ and $Q_i \cap C = q_i$. Moreover, $h(Q_i) = h(Q_{i-1}) + 1$ and $h(q_i) \geqq h(q_{i-1}) + 1$ for $i = 1, \cdots, m$. Q_m is maximal in R (otherwise the process can be continued) consequently (by [12]) $h(Q_0) =$ k.d $R - m$. Also k.dim $C/q \geqq m$ = k.dim R/Q hence if $h(q) \gneqq h(Q)$ then k.dim $C \geqq$ k.dim $C/q + h(q) \gneqq$ k.dim $R/Q + h(Q)$ = k.dim R, a contradiction to k.dim R = tr.$\mathrm{d}_F(C)$. Thus $h(q) \leqq h(Q)$. Finally k.dim R/Q = tr.$\mathrm{d}_F Z(R/Q) \geqq$ k.dim $C/q \geqq m$ = k.dim R/Q, hence k.dim R/Q = k.dim C/q.

PROPOSITION 2.5. *Let $R = F\{x_1, \cdots, x_k\}$ be an affine prime* p.i. *ring*, k.d(R) = 3. *Then, for every maximal ideal m in $Z(R)$, $h(m) = 3$.*

PROOF. Let $m \lhd Z(R) \equiv C$ be a maximal ideal. We have to show that $h(m) \gneqq 2$ ($h(m) \leqq \mathrm{tr}_F C = \mathrm{k.d}(R) = 3$ [7, p. 178]). Indeed, if $h(m) = 1$ then R_m is finite over C_m [1], hence for every prime P in R, $P \cap C = m$, we have $h(P) = 1$. By [12, theorem 4], $1 = h(P) = \mathrm{k.d}\, R - \mathrm{k.d}\, R/P = 3 - \mathrm{k.d}\, R/P$ hence $2 = \mathrm{k.d}\, R/P \neq \mathrm{k.d}\, C/m = 0$, a contradiction to Theorem 2.4. If $h(m) = 2$, then m is contractable from R (e.g. [1]). Let P be some maximal prime to satisfy $P \cap C = m$, enough to show that $\mathrm{k.d}(R/P) \neq \mathrm{k.d}\, C/m = 0$ and get a contradiction. If $h(P) = 1$ we are done as above. Let $0 \neq Q \subsetneqq P$, prime ideal in R with $Q \cap C = q$, $h(q) = 1$. Then since R_q is finite over C_q by [1], Q is maximal to satisfy $Q \cap C = q$, thus, since $h(m) = h(q) + 1$, $(R/Q)_m$ is a G-ring and by Lemma 2.3, $h(P) = h(Q) + 1 = 2$, hence $\mathrm{k.d}(R/P) = \mathrm{k.d}\, R - h(P) = 3 - 2 = 1$ and done. The existence of such Q is granted since either m contains only finitely many height one primes and then R_m is a G-ring and by Lemma 2.1 any $0 \neq Q \subsetneqq P$, Q prime, will do. Or, there are infinitely many q_α's, $q_\alpha \lhd C$, prime, $h(q_\alpha) = 1$, $q_\alpha \subsetneqq m$. Hence there exists $q_\beta = q \subsetneqq m$, $Q \lhd R$, prime with $Q \cap C = q$, $\mathrm{p.i.d}(Q) = n = \mathrm{p.i.d}(R)$. Then by [12] there exists $P' \lhd R$ prime, $P' \supsetneqq Q$ and $P' \cap C = m$, and we take $P \supset P'$ maximal to satisfy $P \cap C = m$.

REMARK 2.6. We don't know if the previous proposition is true for R, an affine prime p.i. ring with $\mathrm{k.d}(R) > 3$.

LEMMA 2.7. *Let $A \subset Z(B)$, $B = F\{x_1, \cdots, x_k\}$ a* p.i. *ring. Let $Q \lhd B$, a prime ideal and $q = Q \cap A$. Suppose that* $\mathrm{k.d}\, A/q \lneqq \mathrm{k.d}\, B/Q$. *Then there are infinitely many prime ideals, Q_α's, $Q_\alpha \supset Q$, $h(Q_\alpha) = h(Q) + 1$ and $Q_\alpha \cap A = q$ for all α.*

PROOF. Let $A_q/q_q = K \subset K\{\bar{x}_1, \cdots, \bar{x}_k\} = B_q/Q_q \equiv \bar{B}$. Then $\mathrm{k.d}(\bar{B}) \geqq 1$. Let $\bar{V} \lhd \bar{B}$, prime ideal, $h(\bar{V}) = 1$. Then $\bar{V} \cap K = \{0\}$. Thus if V denotes its preimage in B, $h(V) = h(Q) + 1$, $V \supset Q$ and $V \cap A = q$. Finally, there are infinitely many such $\bar{V}$'s, hence infinitely many such V's.

LEMMA 2.8. *Let W, Q be prime ideals of R, $W \subset Q$, satisfying $w \equiv W \cap Z(R) \subsetneqq Q \cap Z(R) \equiv q$ and $h(w) \gneqq h(W)$. Then there exists V, Q', prime ideals of R, $V \subset Q'$, satisfying $v \equiv V \cap Z(R) \subsetneqq Q' \cap Z(R) \equiv q$, $h(V) = h(W) + 1$ and $h(v) \geqq h(V)$.*

PROOF. Let $R_0 \equiv R/W$, $Z_0 \equiv Z(R)/w$, $q_0 = q/w$, $Q_0 = Q/W$. We may assume that Q_0 is minimal over q_0. If $h(Q) \gneqq h(W) + 1$ let $W \subsetneqq V \subsetneqq Q$ and $h(V) = h(W) + 1$, V prime. Then by the minimality of Q_0, $v = V \cap Z(R) \subsetneqq q$ and V will do where $Q' = Q$. Suppose therefore that $h(Q) = h(W) + 1$. Then $h(Q) = h(W) + 1 \lneqq h(w) + 1 \leqq h(q)$. Hence

$$\text{k.d}\, Z(R)/q \leqq \text{k.d}\, Z(R) - h(q) \lneqq \text{k.d}\, R - h(Q) = \text{k.d}\, R/Q,$$

thus $\text{k.d}\, Z_0/q_0 \lneqq k.d\, R_0/Q_0$. By Lemma 2.7 there are infinitely many Q_α's, $Q_\alpha \supset Q_0$, $h(Q_\alpha) = h(Q_0) + 1$ for all α. Let $T(Q_0) \subset T(R_0)$ be a prime ideal, $h(T(Q_0)) = 1$ and $T(Q_0) \cap R_0 = Q_0$ (e.g., [12]). Now $R_0/Q_0 \subset T(R_0)/T(Q_0)$, they have the same Krull dimension and by [7, p. 110] again, there exists $Q_\beta \supset Q_0$, prime, $T(Q_\beta) \supset T(Q_0)$, prime in $T(R_0)$, $h(T(Q_\beta)) = h(T(Q_0)) + 1 = 2$ and $T(Q_\beta) \cap R_0 = Q_\beta$. Let $0 \neq T(V_0) \subsetneqq T(Q_\beta)$, prime, such that $T(V_0) \not\supset q_0$ (we have such since $T(R_0)$ is noetherian, $h(T(Q_\beta)) = 2$ and the principal ideal Theorem is true in $T(R_0)$). Let $V_0 = T(V_0) \cap R_0$, $V_0 = V/W$. Then $h(V_0) = h(Q_\beta) - 1 = 1$ thus $h(V) = h(W) + 1$, $v = V \cap Z \subsetneqq q$, $h(v) \geqq h(w) \geqq h(W) + 1 = h(V)$ and $Q' \equiv$ the preimage of Q_β in R.

Corollary 2.9. *With the conditions of the previous Lemma, after a finite number of steps, there exists Q', V prime ideals of R, $Q' \supset V$, $V \cap Z(R) \equiv v \subsetneqq Q' \cap Z(R) \equiv q$ and $h(v) = h(V)$.*

Theorem 2.10. *Let $R = F\{x_1, \cdots, x_k\}$ be an affine prime* p.i. *ring; $q \lhd Z(R)$, prime, $Q \lhd R$, prime and $Q \cap Z(R) = q$. Then $h(q) \geqq h(Q)$.*

Proof. We prove it via induction on $h(q)$. The case $h(q) = 1$ is easy since R_q is a finite module over $Z(R)_q$ ([1]) hence $h(Q) = 1 = h(q)$.

In order to prove the general case, it suffices to show that if P is some prime ideal maximal with respect to $P \cap Z(R) = q$, then $h(q) = h(P)$. Indeed, $\text{k.d}\, Z(R) \geqq \text{k.d}\, Z(R)/q + h(q) = h(P) + \text{k.d}\, R/P = \text{k.d}(R) = \text{k.d}\, Z(R)$. Thus $\text{k.d}\, Z(R) = h(q) + \text{k.d}\, Z(R)/q$. Now, if M is any other prime maximal with respect to $M \cap Z(R) = q$ then $h(q) = \text{k.d}\, Z(R) - \text{k.d}\, Z(R)/q =$ (by Theorem 2.4) $\text{k.d}\, R - \text{k.d}\, R/M = h(M)$.

Following Corollary 2.9 let $Q' \supset V$ be prime ideals of R, $V \cap Z(R) \equiv v \subsetneqq Q' \cap Z(R) \equiv q$ and $h(v) = h(V)$. We may assume that V is maximal to satisfy all these properties. Let $R_0 = R/V$, $Q_0' = Q'/V$, $q_0 = q/v$. We may assume that Q_0' is minimal over q_0. If $h(Q_0') \gneqq 1$ let $V \subsetneqq W \subsetneqq Q'$, prime ideal in R, $h(W) = h(V) + 1$. Then $W \cap Z(R) \equiv w \subsetneqq q$ by the minimality of Q_0', $h(w) \geqq h(v) + 1 = h(V) + 1 = h(W)$. Thus by Corollary 2.9 we reach after a finite step a contradiction to the maximality of V. Hence $h(Q_0') = 1$. If $h(q_0) \gneqq 1$ then by the same argument as in Lemma 2.8, $\text{k.d}\, Z_0/q_0 \lneqq \text{k.d}\, R_0/Q_0'$ hence there exists $V \subsetneqq W \subsetneqq Q''$ with $h(W) = h(V) + 1$, W, Q'' are prime ideals of R, $W \cap Z(R) \equiv w \subsetneqq Q'' \cap Z(R) \equiv q$ and $h(w) \geqq h(v) + 1 = h(V) + 1 = h(W)$ and by Corollary 2.9 we reach again, after finitely many steps, a contradiction to the maximality of V. Thus $h(q_0) = 1$. Now, by Lemma 2.3, Q' is maximal to satisfy $Q' \cap Z(R) = q$, hence by Theorem 2.4, $h(q) \leqq h(Q')$. But $h(q) \geqq h(v)+1 = h(V)+1 = h(Q')$ and consequently $h(q)=h(Q')$. The proof now is complete by the remarks at the beginning.

We now reach one of our main results:

THEOREM 2.11. *Let* $R = F\{x_1,\cdots,x_k\}$ *be a prime affine* p.i. *ring. Let* $q \lhd Z(R)$ *be a prime ideal,* $Q \lhd R$ *be a prime ideal maximal with respect to* $Q \cap Z(R) = q$. *Then*

(1) $\mathrm{k.d}\, Z(R)/q = \mathrm{k.d}\, R/Q$,

(2) $h(q)=h(Q)$,

(3) $\mathrm{k.d}\, Z(R)/q + h(q) = \mathrm{k.d}\, Z(R)$.

PROOF. (1) and (2) are valid by Theorem 2.4 and Theorem 2.10. To prove (3) we observe that by [12] $\mathrm{k.d}\, R = \mathrm{k.d}\, R/Q + h(Q)$, but $\mathrm{k.d}\, Z(R) \geqq \mathrm{k.d}\, Z(R)/q + h(q) = \mathrm{k.d}\, R/Q + h(Q) = \mathrm{k.d}(R)$ by (2) and (3). Thus $\mathrm{k.d}\, R = \mathrm{tr.d}\, Z(R) \geqq \mathrm{k.d}\, Z(R) \geqq \mathrm{k.d}\, R$, hence $\mathrm{k.d}\, Z(R) = \mathrm{k.d}\, Z(R)/q + h(q)$ by comparing the previous inequalities.

COROLLARY 2.12. *Let* R_S *be a* G*-ring* (R, S *as before*), *and* $Q \lhd R_S$ *prime ideal of* R_S. *Let* $q = Q \cap Z(R_S)$. *Then* $h(q)=h(Q)$.

PROOF. By Lemma 2.1 we get that Q is isolated over q. Let $(q_0)_S = q$, $(Q_0)_S = Q$. Then $Q_0 \subset R$ is a prime ideal which is isolated over q_0, hence by Theorem 2.11, $h(q_0)=h(Q_0)$ and consequently $h(q)=h(Q)$.

REMARK 2.13. The condition $\mathrm{k.d}\, Z(R) = \mathrm{k.d}\, Z(R)/q + h(q)$ is a necessary condition for a prime ideal q in $Z(R)$ to be contractable from R. This condition, though, is not sufficient since in the example of [9] $h(q)=5$, q maximal in $Z(R)$, $\mathrm{k.d}(R)=5$ and q is not contractable.

§3. G-Rings of the form R_S

The structure of G-rings of the form R_S, where $R = F\{x_1,\cdots,x_k\}$, prime affine p.i., $S \subset Z(R)$ a multiplicative closed set, is obtained. A key result is the following:

PROPOSITION 3.1. *Let* R_S *be a* G*-ring and* $p \lhd Z(R_S)$ *is a maximal ideal. Then* p *is a finitely generated ideal.*

PROOF. Since $Z(R_S) = Z(R)_S$ then $p = p_{0_S}$ and $p_p = p_{0_{p_0}}$ where $p_0 \lhd Z(R)$ is prime, then $R_{p_0} = (R_S)_p$. We shall show that $p_{0_{p_0}}$ is a f.g. ideal in $Z(R_{p_0})$ and then since $Z(R_S)$ contains only finite number of maximals, we get that p is a f.g. ideal.

We construct a commutative affine F-algebra L, $L \subset Z(R_{p_0})$ with the property that if $m = p_{0_{p_0}} \cap L$ and $Q \lhd R_m$ is prime with $Q \cap L = m$ then Q is isolated over m. The idea is fairly simple but requires a lengthy formal procedure. Let

$$q_0 = 0 \subsetneqq q_1 \subsetneqq q_2 \subsetneqq \cdots \subsetneqq q_{m_0} = p_0 \subsetneqq q_{m_0+1} \subsetneqq \cdots \subsetneqq q_d$$

be a maximal chain of primes in $Z(R)$ with $m_0 = h(p_0)$, $d - m_0 = \mathrm{k.d}\, Z(R)/p_0$, $d = \mathrm{k.d}\, Z(R) = \mathrm{k.d}\, R$ (this can be done since $h(p_0) + \mathrm{k.d}\, Z(R)/p_0 = \mathrm{k.d}\, Z(R)$ by Theorem 2.11). Let $x_i \in q_{i+1} - q_i$ for $i = 0, \cdots, d-1$ and $D = F[x_0, \cdots, x_{d-1}]$. Then $\{q_i \cap D\}_{i=1}^d$ is a proper chain of primes in D. Moreover, $\mathrm{k.d}\, Z(R) = \mathrm{tr.d}\, Z(R) \geqq \mathrm{k.d}\, D \geqq d = \mathrm{k.d}\, Z(R)$ hence $\mathrm{k.d}\, D = \mathrm{k.d}\, Z(R)$ and by the catenary property of D, $h(q_i \cap D) = h(q_i)$ and $\mathrm{k.d}\, D/D \cap q_i = \mathrm{k.d}\, Z(R)/q_i$ for $i = 0, \cdots, d$. One observes that if E is an affine with $D \subset E \subset Z(R)$ then E satisfies the same height and k.d equalities for $\{E \cap q_i\}$ as D does. Let $\mathrm{rad}((p_0 \cap D)R) = T_1 \cap \cdots \cap T_l$, T_i are primes for $i = 1, \cdots, l$, assume that for $i = 1, \cdots, w$, $w \leqq l$, $p_0 \not\subset T_i \cap Z(R)$ and $p_0 \cap D = T_i \cap D$. We term such T_i, $i = 1, \cdots, w$, a "bad" prime. Let $s_1 \in p_0 \backslash (T_1 \cap Z(R)) \cup \cdots \cup (T_w \cap Z(R))$ and $D_1 \equiv D[s_1]$, then $\mathrm{rad}((p_0 \cap D_1)R) = Q_1 \cap \cdots \cap Q_r$ and assume that Q_i are "bad" with respect to $(p_0 \cap D_1)$ for $i = 1, \cdots, t$, $t \leqq r$. But $\mathrm{rad}((p_0 \cap D)R) \subseteq \mathrm{rad}((p_0 \cap D_1)R)$, then if Q_i is "bad" then either there exists a bad T_j with $Q_i \supset T_j$ and since $s_1 \in p_0 \cap D_1 = Q_i \cap D_1$ and $s_1 \notin T_j$ thus $Q_i \neq T_j$, or T_j is good, but

$$p_0 \cap D \subseteq T_j \cap D \subseteq Q_i \cap D = (Q_i \cap D_1) \cap D = (p_0 \cap D_1) \cap D = p_0 \cap D,$$

thus $T_j \cap D = p_0 \cap D$. Now, T_j being good implies $p_0 \subseteq T_j \cap Z(R)$ hence $p_0 \subseteq T_j \cap Z(R) \subseteq Q_i \cap Z(R)$, a contradiction.

Repeating the argument several times we must stop since $\mathrm{k.d}(R) < \infty$. Consequently there exists $E = D[s_1, \cdots, s_v] \subset Z(R)$, affine with $h(q_i \cap E) = h(q_i)$, $\mathrm{k.d}\, E/q_i \cap E = \mathrm{k.d}\, Z(R)/q_i$. Most importantly, let $\mathrm{rad}((p_0 \cap E)R) = W_1 \cap \cdots \cap W_t$ then if $W_j \cap E = p_0 \cap E$ then $W_j \cap Z(R) \supseteq p_0$ (observe that if p_0 is maximal in $Z(R)$, we are done, since the isolation is granted by Lemma 2.1).

Let W_j be termed "bad" (again) if $W_j \cap Z(R) \supsetneqq p_0$ and $w_j \cap E = p_0 \cap E$ and let $W_1, \cdots, W_n$ be the "bad" primes with $n \leqq t$. Choose $y \in (\bigcap_{i=1}^n W_i \cap Z(R) \backslash p_0$ and let $E_1 = E[y, 1/y] \subset Z(R)[1/y] \subset Z(R)_{p_0}$. We have that $\mathrm{k.d}\, E_1 = d$, $h(p_0[1/y] \cap E_1) = h(p_0 \cap E) = h(p_0)$ and hence $\mathrm{k.d}(E_1/E_1 \cap p_0[1/y]) = \mathrm{k.d}(Z(R)/p_0)$ by the catenary properties of E_1. Also $\mathrm{rad}((p_0 \cap E)R[1/y]) = \bigcap_i W_{r_i}[1/y]$ with $y \notin W_{r_i}$, $\{r_i\} \subset \{1, \cdots, t\}$. Suppose $W_{r_j}[1/y]$ is a "bad" prime in $R[1/y]$ then $W_{r_j}[1/y] \cap E_1 = p_0[1/y] \cap E_1$ hence $W_{r_j} \cap E = p_0 \cap E$. If $W_{r_j} \cap Z(R) \supsetneqq p_0$ then $y \in W_{r_j}$, a contradiction. Hence if $W_{r_j}[1/y] \cap E_1 = p_0[1/y] \cap E_1 \equiv m$ then $W_{r_j}[1/y] \cap Z(R[1/y]) = p_0[1/y]$, but

$(W_{r_j})_{p_0}$ is isolated over $p_{0_{p_0}}$ hence $W_{r_j}[1/y]$ is isolated over $p_0[1/y]$ and by Corollary 2.12 $h(W_{r_j}[1/y]) = h(p_0[1/y]) = h(p_0[1/y] \cap E_1)$. Let $V \lhd R[1/y]$, prime with $V \cap E_1 = m$; we show that V is isolated over m. Indeed, $V \supset W_{r_j}[1/y]$ for some j,

$$\text{k.d}(R[1/y]/V) = \text{tr.d}_F Z(R[1/y]/V) \geqq \text{k.d}(E_1/m) = \text{k.d}(E_1) - h(m)$$
$$= \text{k.d}\, R[1/y] - h(p_0[1/y]) = \text{k.d}\, R[1/y] - h(W_{r_j}[1/y]),$$

a contradiction unless $V = W_{r_j}[1/y]$. We take now $L \equiv E_1$.

Let $\pi \in p_0$ with $\text{rad}(\pi Z(R_{p_0})) = p_{0_{p_0}}$ (Lemma 2.1). We may assume that $\pi \in L$ (extend L otherwise). We finally show that p_0 is a f.g. ideal. We have the following inclusions: $L_m \subset Z(R)_m \subset R_m$, and $\Lambda \equiv L_m / \pi R_m \cap L_m \subset Z(R_m)/\pi Z(R_m) \subset R_m/\pi R_m$. Λ is noetherian $R_m/\pi R_m = \Lambda\{x_1, \cdots, x_k, 1/y\}$ and $\text{k.d}\, R_m/\pi R_m = 0$. Hence, by [7, p. 122] $R_m/\pi R_m$ is artinian, hence $p_{0_m}^t \subseteq Z(R)_m$ for some t hence $m_m/\pi Z(R_m) \cap L_m$ is the only prime in Λ and is contracted from $R_m/\pi R_m$. Consequently Λ is artinian and by [7, p. 152, th. 3] $R_m/\pi R_m$ is a finite Λ-module. But then $Z(R_m)/\pi Z(R_m)$ being artinian implies that p_{0_m} is a f.g. ideal. Q.E.D.

THEOREM 3.2. *Let $R = F\{x_1, \cdots, x_k\}$ be a prime affine* p.i. *ring, $0 \notin S \subset Z(R)$ a multiplicative closed set. Suppose that R_S is a G-ring. Then*

(1) *$Z(R_s)$, R_s, have a finite number of prime ideals.*

(2) *Each $q \lhd Z(R_s)$, prime, is contracted in an isolated fashion from R_S.*

(3) $\text{k.d}(R_s) \leqq \text{p.i.d}(R_s) = n$.

(4) *For each $q \lhd Z(R_s)$, prime, q_q is finitely generated and if q is maximal, q is finitely generated.*

(5) *$\overline{Z(R_s)}$ is a prüfer ring, where $\overline{Z(R_s)}$ is the normalization of $Z(R_s)$.*

(6) *Each finitely generated ideal in $\overline{Z(R_s)}$ is generated by $n+1$ (or less) elements.*

PROOF. (1), (2) are consequences of Lemma 2.1. (4) is true by Proposition 3.1 and (5) is a consequence of (4) and Theorem 1.1. (6) is true by [2, p. 453]. To prove (3) we need the following observation. Let $P_1 \subsetneqq P_2 \subsetneqq P_3$ be prime ideals in R, $h(P_3) = h(P_2) + 1$, $\text{p.i.d}(P_1) = \text{p.i.d}(P_2)$, then there are infinitely many primes between P_1 and P_3. Indeed, without loss of generality we assume that $P_1 = 0$, $h(P_2) = 1$. Let $T(P_2) \lhd T(R)$, prime with $T(P_2) \cap R = P_2$, then since $\text{p.i.d}(P_2) = \text{p.i.d}(R)$ there exists $T(P_3) \lhd T(R)$ prime, $T(P_3) \supset T(P_2)$ and $T(P_3) \cap R = P_3$. Now $h(T(P_3)) = 2$ and there are (by the principal ideal theorem) infinitely many height one primes under $T(P_3)$ and all but finite contracts to infinitely many height one primes in R. Going back to R_s and invoking (1), we see that if $P \subsetneqq Q$ prime ideals of R_s, $h(Q)=h(P)+1$ then either $\text{p.i.d}(P)>\text{p.i.d}(Q)$ or Q is maximal. In each case we get that $\text{k.d}(R_s) \leqq \text{p.i.d}(R_s)$.

REMARK 3.3. Unlike the situation in the noetherian commutative case, in general R_S being a G-ring doesn't imply that $\text{k.d}\, R_s = 1$ although a bound is achieved by Theorem 3.2 (3). The following example, which is adapted from an example due to G. Bergman (private communication), illustrates this phenomenon.

EXAMPLE 3.4. Let s, x be commutative variables. Let

$$R = \begin{pmatrix} k[s]+xk[s,s^{-1},x], & k[s,s^{-1}]+xk[s,s^{-1},x] \\ xk[s,s^{-1},x], & k[s,s^{-1}]+xk[s,s^{-1},x] \end{pmatrix},$$

R is a prime p.i. affine ring, $\text{k.d}(R)=2$, the generators are sI, se_{12}, $s^{-1}e_{22}$, e_{11}, xe_{21} (I is the identity matrix) and R is not noetherian since $\begin{pmatrix} k[s], & k[s,s^{-1}] \\ 0, & k[s,s^{-1}] \end{pmatrix}$ is a homomorphic image of R and is right but not left noetherian. $Z(R)= k[s]+xk[s,s^{-1},x]$ hence, since $k[s,s^{-1},x]$ is normal and $Z(R)$ is integrally closed in $k[s,s^{-1},x]$, we have that $Z(R)$ is normal. Also conductor $(Z(R), k[s,s^{-1},x])=xk[s,s^{-1},x]$. Observe that $sk[s]+xk[s,s^{-1},x]$ is a maximal ideal in $Z(R)$ which is *not* contracted from $k[s,s^{-1},x]$ since $s^{-1}\in k[s,s^{-1},x]$. But

$$T(R)=\begin{pmatrix} k[s,s^{-1},x], & k[s,s^{-1},x] \\ xk[s,s^{-1},x], & k[s,s^{-1},x] \end{pmatrix}$$

and $Z(T(R))=k[s,s^{-1},x]$. The primes in R contracting to $p=xk[s,s^{-1},x]$ — a prime of height one in $Z(R)$ — are

$$\begin{pmatrix} xk[s,s^{-1},x], & k[s,s^{-1}]+xk[s,s^{-1},x] \\ xk[s,s^{-1},x], & k[s,s^{-1}]+xk[s,s^{-1},x] \end{pmatrix}$$

and

$$\begin{pmatrix} k[s]+xk[s,s^{-1},x], & k[s,s^{-1}]+xk[s,s^{-1},x] \\ xk[s,s^{-1},x], & xk[s,s^{-1},x] \end{pmatrix}.$$

One obtains that R_q is a G-ring where $q=sk[s]+xk[s,s^{-1},x]$, since p_q, q_q are the only non-zero primes which are contained in $Z(R_q)=Z(R)_q$. We also have that $\text{k.d}(R_q)=\text{p.i.d}(R_q)=2$ and $Z(R)_q$ is a valuation ring.

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DEPARTMENT OF MATHEMATICS
HAIFA UNIVERSITY
HAIFA, ISRAEL